EXPERIENCE-BUILDING IN THE TEACHING OF GEOMETRY

In this school we have ventured to deviate from the traditional teaching of geometry in several ways. In the change from the time-honored course, some material has been eliminated. The sequence of subject matter has been greatly altered. The traditional Greek division of geometry into five books has been abandoned. In addition

PUPILS LAYING OUT A RIGHT ANGLE BY MEANS OF THE OPTICAL SQUARE

to the compasses and straight-edge to which the constructions of geometry have been limited since the time of Plato, use is made of the other instruments that are employed in geometrical constructions in modern times. The subject has been simplified by assuming, from construction or observation, some of the difficult theorems at the beginning of the subject that it has been the custom to have pupils prove. Instead of the common method in which pupils merely memorize the proofs of theorems as they are given in the book, the suggestive method is employed, in which mere memorizing is made impossible and pupils are taught to think things out for themselves. The barriers between geometry and other subjects, such as algebra

and trigonometry, are broken down, and some degree of correlation of all these subjects secured. And last, but not least in importance, the content of the subject matter is vitally affected in the effort to relate the subject to the real and practical experiences of the pupils. It is with this last phase of the teaching of geometry that this report is to deal.

The attempt to relate geometry in the secondary schools to the concrete or practical experiences of the pupils, which has in the last few years attained the magnitude of a national movement, has resulted from the newer view of the nature and aim of true education and from the recognition of certain fundamental laws of psychology in harmony with this view. In true education the mere acquisition of knowledge is not the end. Rather, knowledge should be acquired as the result of purposeful action, and should in turn become a means to action. From this view of the place of knowledge in education two corollaries follow: First, true knowledge on the part of an individual must grow out of his own experience. Psychologically, knowledge to be real must have as basis a body of clearly defined mental imagery; this body of imagery in turn must result from some form of experience. It is out of this body of imagery that creative imagination is built, which is the basis of all purposeful conduct, all initiative, all originality, all constructive action. Second, the complete educative process includes not only the acquisition of knowledge but also its functioning in action. Knowledge is not complete until it functions. Psychologically, knowledge to endure must be provided opportunity to function through use. Knowledge not used, and used promptly, tends to atrophy. The violation of these principles in teaching constitutes not only the lost opportunity, or even stupidity, of formal education, but its gross wrong to childhood. Formal education has too persistently lost sight of the real, natural activities and experiences which constitute the very sum of child life and out of which alone true knowledge can grow, and such knowledge as does result is too often a valueless and fleeting thing because it is afforded no opportunity to function in action.

To the traditional teaching of geometry this criticism applies. The three essential psychological steps in the educative process are laying a foundation of experience upon which to build, organizing a body of knowledge out of this experience, and finally applying the resulting knowledge to some kind of practical use in the concrete world.

Of these three steps, in the traditional teaching of geometry the first and last have been omitted and only a step corresponding to the middle one supplied. In the first place, the geometrical concepts and principles with which the pupil is to struggle are not thoroughly grounded in his personal experience. They are largely intellectual abstractions. In beginning geometry the pupil is immediately introduced to two difficulties at once—to reasoning about abstract things beyond his mental horizon, and to struggling in the attempt to master

PUPILS MEASURING HEIGHT OF BUILDING BY MEANS OF GEOMETRIC SQUARE

the intricacies of an exacting logical demonstration. And again, once the pupil has gained a mastery of these difficulties, if ever, he is not allowed opportunity to put the knowledge acquired to use in the solution of real practical problems or in purposeful action of any kind.

In teaching geometry in this school we have attempted to supply all three of the steps in the educative process. In the first place, formal demonstrative geometry is preceded by a considerable amount of inductive work, in which, through constructions, measurements, and observations, many of the fundamental concepts and principles of geometry are developed in an objective way. For example, the pupil gets a clear mental image of complementary angles, and discovers the principle that complements of the same or equal angles are equal,

through constructions and practical uses of such angles, before he is introduced to the formal demonstration of theorems in which the use of the principle of complementary angles is involved. In addition to building thus a foundation of concrete experience and imagery upon which later to base the work of formal demonstrations, the pupil becomes familiar with the various drawing instruments and their uses in the construction of some of the principal geometrical figures which later are involved in the theorems.

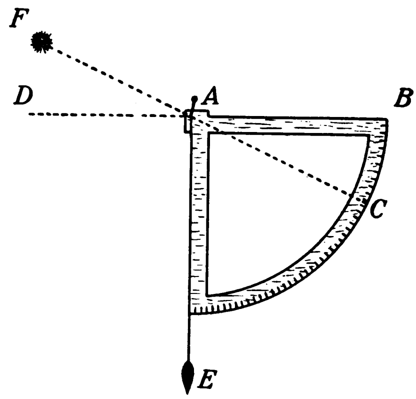
Again, many of the theorems of formal geometry are approached concretely, so that a body of experience and imagery is built about them before their logical demonstration is undertaken. For example, the theorem that the sum of the angles of a triangle equals one straight angle is shown objectively in a number of ways. One way is to cut out a paper triangle, then tear off the three corners and place them adjacent, so that they form a straight angle. This is actual physical addition of the angles. Another method is to turn a ruler through the three angles of a triangle in succession, and note that the ruler has just been reversed in direction, or turned through a straight angle. A third is to measure the angles of a triangle with a protractor, and find their sum arithmetically. Similarly, the proofs of theorems on the congruence of triangles are preceded by geometrical constructions and physical superpositions, which supply adequate mental imagery for the logical demonstrations that follow. In addition to such methods as these for grounding the knowledge of geometry on the personal experiences of the individual pupils, it will readily be seen that applications of many of the theorems of geometry to practical problems, as will be described in the subsequent part of this report, surround those theorems by a body of imagery that gives clearness to their meanings.

Applying the general principle that knowledge of geometry, to be real and permanent, must be afforded opportunities to function through immediate use, we have attempted, with considerable success, to make applications of the constructions and theorems of geometry to practical problems, usually immediately following the demonstration of them. While direct practical applications of some theorems have not been found, suitable, and in some cases excellent, practical problems have been found for applying many of the theorems of geometry.

These applied problems have been selected from many fields of

human activity, such as surveying, bridge-building, carpentry, designing, architecture, physics, astronomy, the construction and use of various measuring instruments, etc. The following miscellaneous typical problems will serve to show their character and also their relations to the standard theorems of geometry.

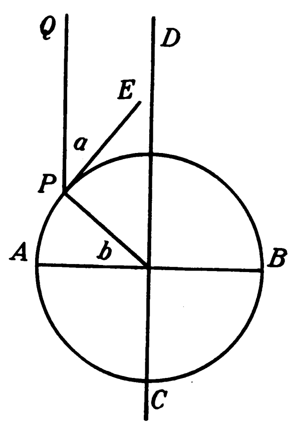
PROBLEM.—The angle FAD of elevation of the sun may be obtained as follows: A quadrant is held in a vertical position so that a plumb line AE, fastened to a pin at the vertex A, falls upon 90°. The pin casts a shadow on the scale at C. Show that the angle of elevation is obtained by reading the number of degrees on the quadrant from B to C.



The proof is based upon the theorem:

If two straight lines intersect, the vertical angles formed are equal.

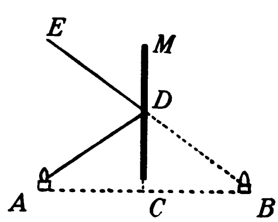
PROBLEM.—The latitude of a point on the earth may be found by observing the altitude of the North Star. Prove that the latitude of the observer equals the altitude of the North Star; i. e., $a=b$. (AB is the equator, and CD the axis of the earth. If the observer is at P, the direction PQ to the North Star is parallel to the axis, because of the great distance of the star from the earth.)



The proof is based upon the theorem:

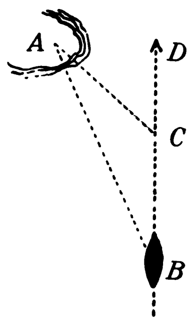
If two angles have their sides perpendicular, each to each, and both are acute, they are equal.

PROBLEM.—Every one is familiar with the fact that if an object is placed before a plane mirror, its image appears to be as far behind the mirror as the object is in front of it. Prove that this must always be so. (M is an edge view of the mirror. The image of the object A is B. Light from A strikes the mirror at D, and is reflected to the eye at E. The mind projects the ray ED to B. The angle at which light is reflected from a mirror equals the angle at which it strikes. Prove $AC=BC$).



The proof is based upon the theorem:

If two triangles have two angles and a side of one equal respectively to two angles and the corresponding side of the other, the triangles are congruent.

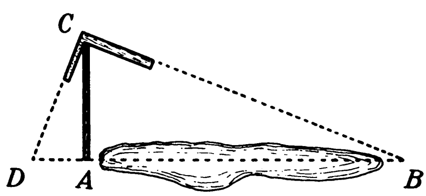


PROBLEM.—A ship, going in the direction BD passes the lighthouse A. From aboard the ship the angle ABD which the lighthouse makes with its course is observed. Later the position C of the ship is noted, from which the lighthouse makes with the ship's course an angle twice as large. Prove that the distance the ship has moved equals its distance from the lighthouse when in the second position.

The proof is based upon the theorems:

The exterior angle of a triangle equals the sum of the two opposite interior angles.

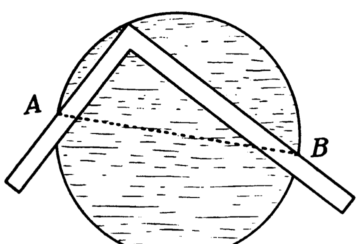
If two angles of a triangle are equal, the sides opposite the equal angles are equal, or the triangle is isosceles.



PROBLEM.—A method used several centuries ago for determining the distance from a point A to an inaccessible point B was to erect a vertical staff AC, place upon this an instrument resembling a carpenter's square, direct one blade toward B, and note the point D on the ground toward which the other blade pointed. Show how to find AB by measuring AC and AD. If $AC = 6$ ft. and $AD = 3$ in., find AB.

The proof is based upon the theorem:

If two triangles are similar, the homologous sides are in proportion.

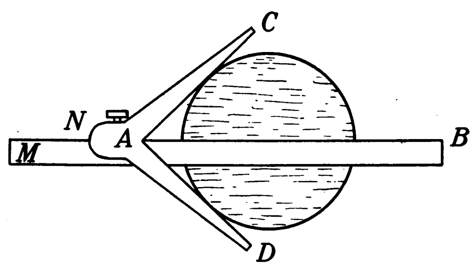


PROBLEM.—Prove that the center of any circular object may be located by means of a carpenter's square, as follows: Lay the square on the object, with the heel at the rim, and mark the points A and B where the blades cross the rim. Now, by placing a blade of the square on A and B, find the middle point of AB. That is the center.

The proof is based upon the theorem:

An inscribed angle of a circle has the same measure as one-half of the intercepted arc.

PROBLEM.—The instrument called a center square is used for locating the centers of circular objects. It consists of a steel bar or blade M, upon which slides an attachment N. The edge AB of the blade M bisects the angle between the two prongs AC and AD of the attachment N. When the center of any circular object is to be found, the instrument is placed so that the prongs AC and AD are tangent to it. Prove that when this is done the edge AB passes over the center of the object.

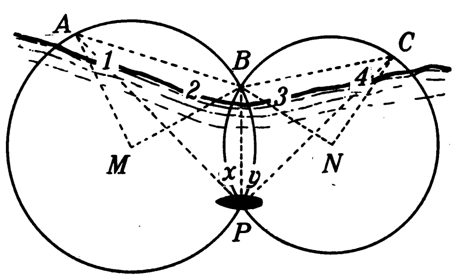


The proof is based upon the theorems:

The locus of points within an angle and equidistant from its sides is the bisector of the angle.

The radius drawn to the point of contact of a tangent is perpendicular to the tangent.

PROBLEM.—An important problem in marine surveying is to determine the position P of a boat from which soundings are being made along a coast. The boat moves from place to place, and it is necessary to locate these positions on the chart. Three stations, A, B, and C, are located on the shore. Angles x and y are observed from the boat. A, B, and C, are located on the chart. P is located on the chart by the intersection of two circles passing through A, B, and C. Show how to locate their centers. Suppose $x = 40°$, $y = 70°$.

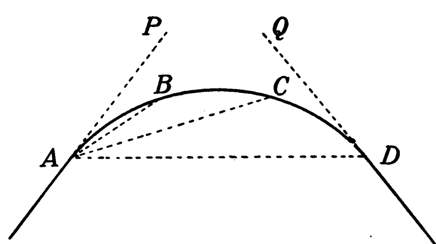


The construction and proof depend upon the theorems:

An inscribed angle has the same measure as one-half of the intercepted arc.

The angles opposite the equal sides of an isosceles triangle are equal.

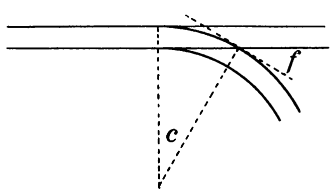
PROBLEM.—In railroad surveying, curves are laid out by turning off equal angles and setting stakes every 100 ft. If the curve begins at A, angle PAB is turned off from the tangent AP, and AB measured 100 ft., then angle BAC is turned off and BC measured 100 ft., and so on until the curve ends in the tangent DQ at D. Since the curve is to be the arc of a circle,



show that angles PAB, BAC, etc., must be made equal, and each equal to one-half of the central angle subtended by a 100-ft. chord.

The proof depends upon the theorem:

An inscribed angle of a circle has the same measure as one-half of the intercepted arc.

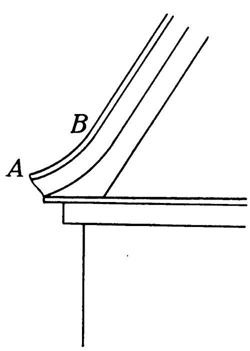


PROBLEM.—In laying out switches on a railroad track, a "frog" is used at the intersection of the rails to allow the flanges of the wheels moving on one rail to cross the other. The angle of the "frog" that must be selected for any place depends upon the central angles of the two tracks. If one track is straight and the other curved, prove that angle f of the "frog" equals the central angle c of the curved track.

The proof depends upon the theorems:

The angle between a tangent and a chord has the same measure as one-half of the intercepted arc.

A radius perpendicular to a chord bisects the subtended arc.

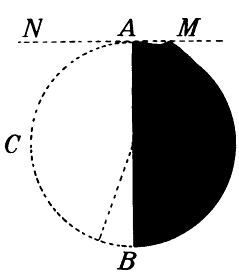


PROBLEM.—In architecture it is required sometimes to draw an easement cornice tangent to the straight or rake cornice at B, and ending at a given point A. Explain the construction, and make such a drawing. (NOTE.—The same construction is used in laying out the easements of stair rails.)

The construction and proof depend upon the theorems:

The perpendicular to a tangent at the point of contact passes through the center of the circle.

The perpendicular bisector of a chord passes through the center of the circle.



PROBLEM.—Galileo measured the heights of the mountains on the moon as follows: ACB was the illuminated half of the moon just as the peak of the mountain M caught the beam NM of the rising or setting sun. He measured the distance AM. Show how he was able, by using the known diameter of the moon, to compute the height of the mountain.

The computation and proof depend upon the theorem:

If from a point without a circle a tangent and secant are drawn, the tangent is a mean proportional between the whole secant and its external segment.

In addition to such miscellaneous applied problems of geometry as those described above, there is a special group of applied problems

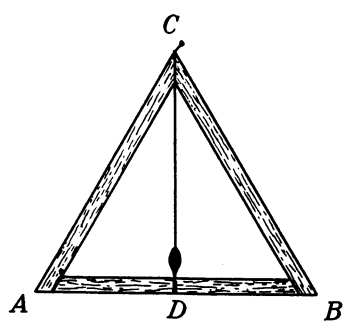
used by our classes in geometry which should be described on account of their exceptional value and interest to boys and girls. The special value and interest are due to the fact that these problems involve activity, involve doing as well as reasoning on the part of the pupils. They relate to the construction and uses of various instruments for measuring, surveying, etc.

For a number of years, the pupils of the geometry classes have made certain simple instruments, and then used them, mostly out of doors, in doing certain practical work. All of the instruments have been made in the manual training room at school, most of them outside of regular school hours.

A number of the instruments made by the pupils are similar to the simple instruments that were used in practical surveying, leveling, etc., centuries ago, before the modern delicate and more complicated ones were invented. While many of them have been superseded in practical life today, they possess special interest in showing how the world's work has been done in the past, and a special value in that each, in its construction and use, involves a simple, direct application of one or more of the standard theorems of geometry.

A description of some of these instruments and of the kinds of practical problems in which they are used, and the theorems of geometry underlying their construction and uses, follow:

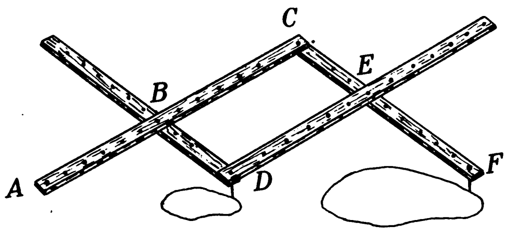
THE PLUMB LEVEL.—This instrument, which was used for leveling before the modern spirit or bubble level was invented, takes its name from the plumb line, which is the significant part of the instrument, just as it was in most of the measuring instruments used in early times. It consists of three boards, *AB*, *AC*, and *BC*, which are fastened together so as to form a triangle. *AC* and *BC* are of the same length. A mark *D* is placed at the middle point of *AB*. A plumb line is suspended from *C*.



When being used the instrument is held with *AB* upon the surface to be leveled. If the plumb hangs directly over the mark *D*, the surface is level.

Since a level object must be horizontal, that is at right angles to vertical, and since the plumb line marks a vertical line, the proof that the surface is level consists of proving that *AB* is perpendicular to *CD*. This is an application of the theorem:

In an isosceles triangle the median to the base is perpendicular to the base.



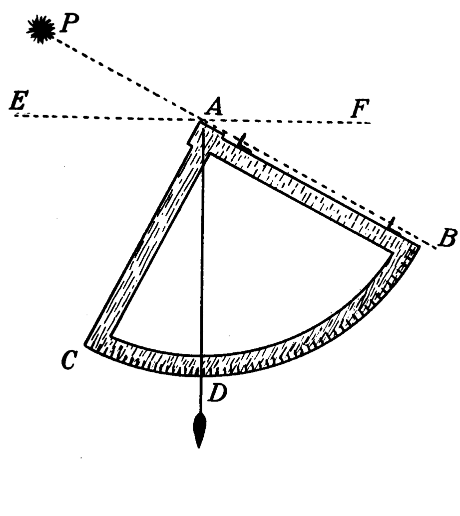
THE PANTAGRAPH.—The pantagraph is an instrument for drawing a plane figure similar to a given plane figure, and is useful for enlarging or reducing maps or designs. It consists of four bars, parallel in pairs, and joined by pivots at B, C, D, and E. A turns on a fixed pivot, and pencils are carried at D and F. BD and DE are adjusted so as to form a parallelogram $BCED$ and so as to make any required ratio $\frac{AB}{AC}$ equal to $\frac{CE}{CF}$.

When used for enlarging a drawing, the pantagraph is placed over the drawing, and when it is turned about the fixed pivot A, the pencil D is moved around the boundary of the drawing. At the same time the pencil F traces an enlarged drawing, similar to the given one.

The use of the instrument requires the proof (1) that A, D, and F are in a straight line, and (2) that the ratio $\frac{AD}{AF}$ remains constant and equal to the given ratio $\frac{AB}{AC}$, so that if D traces a given figure, F will trace a similar figure, the ratio of similitude being the fixed ratio $\frac{AB}{AC}$. The proof depends upon the theorems:

The opposite sides of a parallelogram are equal.

If two triangles have two sides of one proportional to two sides of the other, and the included angles equal, the triangles are similar.



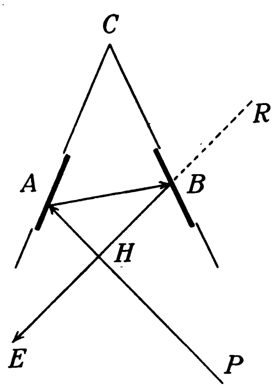
THE QUADRANT.—The quadrant consists of the fourth of a circle, with the arc BC graduated into 90°. The radius AB bears two sights. A plumb line, suspended from a pin at the center A, crosses the arc at D.

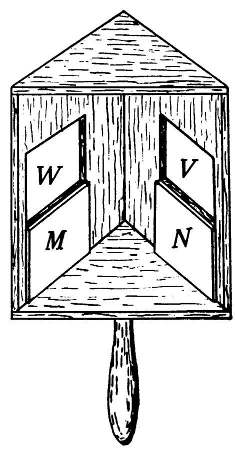
The quadrant is used for determining angles of elevation or depression. Tycho Brahe (1546-1601), a Danish astronomer, used such an instrument for finding the altitudes of stars, i. e., their angular distances above the horizon. Our pupils

use it for finding the altitude of the North Star, in order to determine the latitude of Chicago. They use it also in problems to find the heights of objects in the neighborhood, by measuring the base line and using a table of tangents, or by drawing to scale. The number of out-of-door problems that may be solved by measuring angles of elevation or depression with the quadrant is almost unlimited.

When the angle *EAP* of elevation of an object, as a star *P,* is being measured, the quadrant is held in a vertical plane, and the radius *AB,* by means of the sights, pointed toward *P.* By reading the arc *CD* the number of degrees in angle *EAP* is obtained. Since the plumb line *AD* is at right angles to the horizontal line *EF,* and angle *BAC* is a right angle, the proof depends upon the theorem:

Complements of the same or of equal angles are equal.





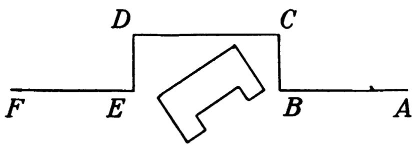
THE OPTICAL SQUARE.—The optical square is an instrument used in field work, such as forestry, for laying out right angles. It consists of a box with triangular top and bottom, and but two side walls, which are set at an angle of exactly 45°. In these walls are cut openings or windows, *W* and *V.* Below the windows, mirrors, *M* and *N,* are fastened against the walls.

When a line is being laid out at right angles to a given line at a given point, the instrument is held in a vertical position at this point, and the observer, looking directly into the box through the open side, turns the box until he can see, through one of the windows, say *V,* an object marking another point in the given line. Then an assistant with a target rod takes a position such that the image of the target appears in the mirror *N* just below or vertically coincident with the object seen through window *V.* The target rod and the position of the observer then mark out the required perpendicular line.

The diagram shows the path of the line of sight through the optical square. Light coming from the object at *P* strikes the mirror at *A,* is reflected to the mirror at *B,* then reflected to the eye of the observer at *E.*

The mind projects the image of the object to R. Then ER is at right angles to AP. Assuming as hypothesis that the angles of reflection and incidence of the light are equal at A and also at B, and that angle C is 45°, the proof depends upon the theorem:

The sum of the angles of a triangle is a straight angle.

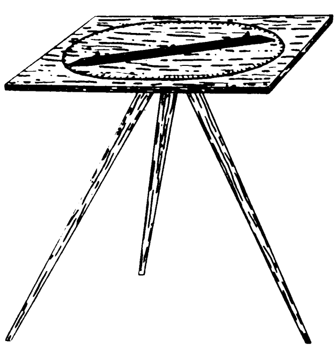


One of the practical uses that we have made of the optical square in out-of-door work has been in the problem to continue a given line AB beyond an obstacle, one of the practical problems frequently encountered in civil engineering. By means of the optical square, the offset BC is run at right angles to AB, then DC is run beyond the obstacle at right angles to BC. Then the offset DE is run equal to BC and at right angles to DC, then EF at right angles to DE. The proof that EF is a prolongation of AB depends upon the theorems:

Two lines perpendicular to the same line are parallel.

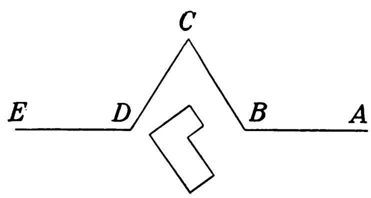
If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

Another out-of-door use that we have made of the optical square is in playing pirates. One geometry class, playing that it is a band of pirates, by using the optical square, a tape line, and target rods, surveys a line with many turns and runs, and at the end of it buries in the ground a silver chest of treasure (possibly a tin can with something in it). A description of this survey prepared by the class is given, possibly the following day, to another class, which re-surveys the line to find the treasure. Boys and girls enjoy this game. The play instinct of pupils of this age is by no means dormant.



THE TRANSIT.—We gave this name to the instrument because of the similarity of its uses to those of the modern engineer's transit. The transit consists of a board mounted horizontally upon a tripod (a converted plane table), the board bearing a pointer revolving over a circle marked off into 360°. The instrument can be used in all kinds of out-of-door problems that involve the measurement or construction of an angle in a horizontal plane.

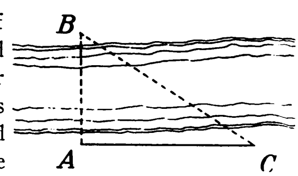
We have used it in playing the game of pirates, where the line surveyed turned at angles of different sizes. We have used it also in the problem of civil engineering to continue a line AB beyond an obstacle by the 60-degree method. The transit is set up at B,



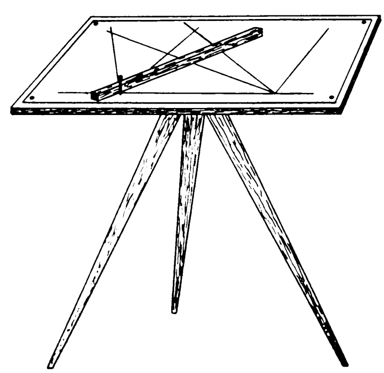
and the line BC run beyond the obstacle, at an angle of 60° with the prolongation of AB through B. Then the transit is set at C, and line CD run equal to BC and making angle BCD 60°. Then the transit is set at D, and DE run at an angle of 60° with the prolongation of CD through D. The proof that DE is a prolongation of AB depends upon the theorems:

The sum of the angles of any triangle equals a straight angle.

The angles opposite the equal sides of an isosceles triangle are equal.

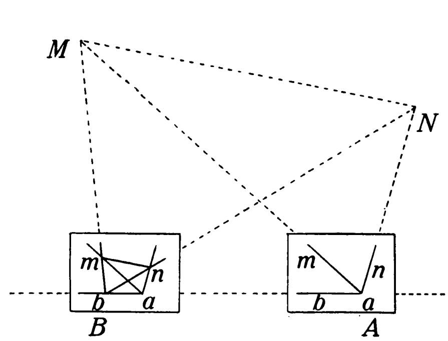


Among other uses that we have made of the transit in out-of-door problems, is to find the width of the lagoon in Lincoln Park, near the school. To determine AB, the operator sets the transit at A. A right angle BAC is turned off with the transit, and a stake set at C. The transit is then removed to C, and the angle ACB measured. The length of AC is found with a tape line. Then AB is computed by use of a table of tangents. It might be found also by drawing the triangle ABC to scale and measuring AB with a ruler.



THE PLANE TABLE.—This instrument consists of a drawing board mounted upon a tripod. A straight-edge is placed upon the board for sighting and drawing lines. It is employed for making maps of small areas, and for determining the distances and directions between inaccessible objects.

One field problem in which we have used the plane table is to find the distance between two points without approaching either of them, such as the distance from the conservatory to the animal house in Lincoln Park. To find the distance from M to N, the operator sets the plane table up at any convenient point A. A sheet of paper is fastened on the board, and a pin stuck through the paper into the board at a point a, directly over A. The ruler is placed against the pin, and a base line ab drawn on the paper toward a second point B. Lines an and am are

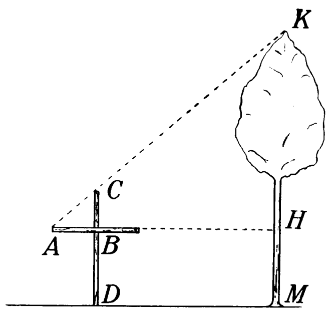


then drawn toward N and M, respectively. The plane table is then removed and set up over point B, so that the line ab on the paper is directly above B and points directly back to the old station A. The pin is then removed to a point b, directly over B, in line ab. Lines bn and bm are then drawn, again toward N and M, respectively. The distance from A to B is measured with a tape line. Lines an and bn meet at n, and am and bm meet at m. Then mn is drawn and measured. Also ab is measured. From these measurements the distance MN is computed by proportion. The proportion is $\frac{MN}{AB} = \frac{mn}{ab}$. The proof of this proportion depends upon the theorems:

The homologous sides of similar triangles are in proportion.

If two triangles have two pairs of sides in proportion and the included angles equal, the triangles are similar.

We have used the plane table also for making maps of small surfaces, such as the ponds in Lincoln Park. The procedure is somewhat similar to that described above. A description of the method may be found in the text book (Stone-Millis).

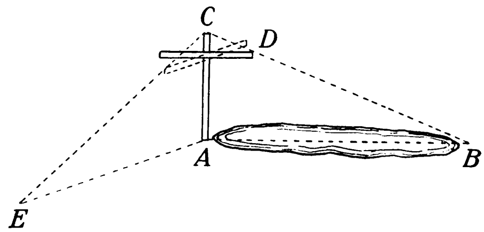


THE CROSS-STAFF.—The cross-staff was used centuries ago in practical surveying for finding heights and distances. It consists of a vertical staff which supports a horizontal cross-piece that may be lowered or raised at will. We have used it to measure heights and distances in various problems out of doors.

When the height of an object MK is being found, the cross-bar AB is raised or lowered until, by sighting along A and C, the points A, C, and K, are brought into a straight line. Then the lengths of AB and BC are noted and the distance AH measured. HK is computed by use of the theorem:

If two triangles are similar, the homologous sides are in proportion.

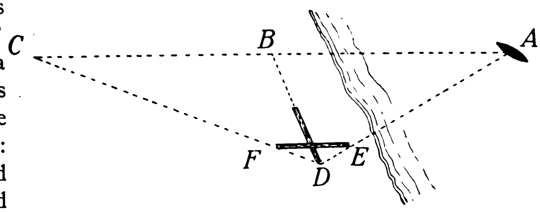
The height BD, which equals HM, is added to the result to obtain MK.



When the distance from a point A to an inaccessible point B is being determined by means of the cross-staff, the cross-bar is raised or lowered until points C, D, and B fall in one line. Then, holding the cross-bar in position, the instrument is revolved about AC as axis, and the point E noted at which the line of sight CD strikes the ground. AE is measured. AB equals AE. The proof is based upon the theorem:

If two right triangles have a leg and acute angle of one equal, respectively, to a leg and acute angle of the other, the triangles are congruent.

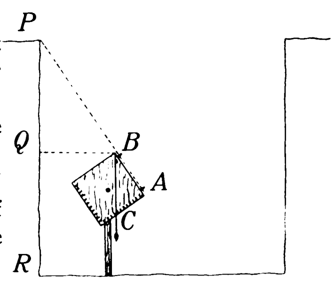
Such a problem as to find the distance AB across a stream, or of a ship from shore, is solved by use of the cross-staff as follows: The instrument is placed with both the staff and



cross-bar in horizontal positions, the staff being pointed in a line parallel to the shore line. The cross-bar is adjusted until D and E fall in line with A. Then an assistant is located at C, directly inshore from A and in line with D and F. BC is measured. AB equals BC. The proof depends upon the theorem above and the following:

Two right triangles having the two legs of one equal to the two legs of the other are congruent.

THE GEOMETRICAL SQUARE.—The geometrical square is one of the instruments most extensively used a few centuries ago for measuring heights, depths, and distances. It consists essentially of a square frame along two adjacent edges of which is marked a scale, and from the opposite corner of which a plumb line is suspended. A pair of sights on another edge aid in sighting the instrument toward a given point.

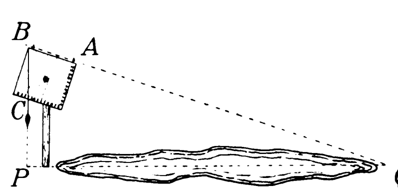


We have used the geometrical square for determining heights and distances of various objects in the neighborhood of the school. One group used it also for determining the depth of a quarry in the suburbs of the city, which an elementary grade wished to know.

When the height PR of an object is to be found, the square is held in a vertical plane, and, by means of the sights A and B, the edge AB is pointed toward P. The plumb line crosses the graduated scale along one of the edges at C. The lengths of the sides of the right triangle thus formed on the instrument are noted. The horizontal distance BQ is measured. Then PQ is computed by proportion. The height of the instrument, which equals QR, is added to the result to obtain PR. The proportion used depends upon the theorem:

If two triangles are similar, the homologous sides are in proportion.

When a horizontal distance PQ is to be determined by use of the geometrical square, the instrument is held directly over P, and the edge AB is sighted toward Q. The readings on the instrument are noted as in the problem above. The height



BP of the instrument above the base line *PQ* must be known or found. Then *PQ* is computed by a proportion, which is determined from similar triangles.

We are positive that the boys and girls in any secondary school would gladly and easily make such instruments as those described above, and that the interest and geometrical insight which would attend the practical use of them in out-of-door problems would more than repay the effort. We unreservedly recommend such work to teachers of geometry.